The Banach-Tarski Paradox and Paradoxical Decomposition

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1 Introduction

In this short research paper, we will introduce the famous Banach-Tarski paradox and related concepts. We will include some easy-to-understand proof sketches, as well as the historical background of these ideas. The technical content of the paper is inspired by [1] and [2].

The Banach-Tarski paradox belongs to a family called paradoxical decomposition, where the gist is that you can decompose some shape in a Euclidean space into finitely many pieces, and then you can obtain some paradoxical outcomes regarding to the volume of the pieces. The existence of most of these paradoxes relies on a controversial axiom of the set theory, namely, the Axiom of Choice, which we will introduce later. Proceeding the Banach-Tarski paradox, there is the Hausdorff paradox; succeeding it, there is the Von Neumann paradox, all of which follow a similar proof technique.

In section 2, we talk about the historical context and the historical figures behind these concepts. In section 3, we formally introduce the paradoxes and the related axioms and notations, as well as prove some selected theorems.

2 Background

In 1901, British mathematician Bertrand Russell discovered the famous Russell's paradox which showed that the naive set theory created by Georg Cantor contained some inner inconsistency. Since then, mathematicians strove to create a consistent, formalized set theory, which led to the Zermelo-Fraenkel set theory in early twentieth century. At the same time, based on the new notion of set, mathematicians developed another new system of theory – the measure theory, a systematic way to assign a number to each suitable subset of a set. Main contributors include Émile Borel, Henri Lebesgue, Johann Radon, and Maurice Fréchet, among others.

Felix Hausdorff, a German mathematician born in 1868, discovered the Hausdorff paradox in 1914, which showed an interesting paradoxical phenomenon based on the new system of set theory and measure theory. Later, Stefan Banach and Alfred Tarski, both Polish by born, proposed the famous Banach-Tarski paradox in 1924, which is a similar but more powerful paradox compared to the Hausdorff paradox. Both Hausdorff and Banach's later life was largely affected the Second World War. Tarski was fortunate enough to escape to the United States and taught Math in UC Berkeley for the rest of his life.

In 1903, a child prodigy called John von Neumann was born in Budapest. Later, he moved to the United States in the thirties and made huge contribution to the early development of Computer Science. In 1929, he proved the John von Neumann paradox, which is a even more powerful version of the Banach-Tarski paradox.

3 Theories

In set theory, Zermelo-Fraenkel set theory, named after mathematicians Ernst Zermelo and Abraham Fraenkel, is an axiomatic system that was proposed in the early twentieth century in order to formulate a theory of sets free of paradoxes such as Russell's paradox. Zermelo-Fraenkel set theory with the Axiom of Choice included is abbreviated **ZFC**, where C stands for "choice", and **ZF** refers to the axioms of Zermelo-Fraenkel set theory with the axiom of choice excluded.

The Axiom of Choice was formulated in 1904 by Ernst Zermelo in order to formalize his proof of the well-ordering theorem. It is a seemingly "very true" statement.

Axiom 1 (Axiom of Choice). *Let $\{S_i\}_{i \in I}$ be an countable collection of nonempty sets, where I is the index set. We can then choose one element from each S_i and form a new set $X = \{s_i\}_{i \in I}$ where $s_i \in S_i$.*

AoC is very intuitive or "obviously true". However, it entails some unexpected paradoxical results. Among them, the simplest one is probably the existence of "non-measurable" sets, i.e., the sets that cannot be assigned a meaningful measure or "volume". Let us see one example.

Theorem 2 (Existence of a non-measurable subset of [0,1]). *Assuming the Axiom of Choice, then there exists a set $X \subseteq [0,1]$ such that the measure of X cannot be defined.*

Proof. We know that $[0,1]$ is a set of real numbers. We now define a relation $\sim$ on $[0,1]$ as $a \sim b$ if there exists $q \in \mathbb{Q}$ with $a + q = b$, for all $a, b \in [0,1]$. That is, two numbers are related if they differ by a rational number. It is easy to show that $\sim$ is an equivalence relation on $[0,1]$. Thus, $\sim$ partitions $[0,1]$ into equivalence classes. In particular, all the rationals inside $[0,1]$ form one equivalence class. We also have that $\sqrt{2}$ and $\sqrt{3}$ are not in the same class since $\sqrt{3} - \sqrt{2}$ is not rational.

Now by Axiom of Choice, we can form a new set X by choosing one representative from each equivalence class. Here is the question, what is the measure of X?

Now consider the set of rational numbers between -1 and 1, i.e., $\mathbb{Q} \cap [-1,1]$. Since rational numbers are enumerable, we can list the elements in the set one by one, as $r_1, r_2, r_3, \cdots$ Let us define a collection of new sets $\{X_i\}_{i \in \mathbb{N}}$ where $X_i = \{x + r_i : x \in X\}$. In particular, we can fix j with $r_j = -1$, and we have $X_j \subseteq [-1,0]$; we can fix k with $r_k = 1$, and we have $X_k \subseteq [1,2]$. Now note that $X_i \subseteq [-1,2]$ for any $i \in \mathbb{N}$. Thus if we take the union of all X_i, i.e., $\bigcup\{X_i\}_{i \in \mathbb{N}}$, the measure of this union must be less or equal to 3, since the measure of $[-1,2]$ is 3 (here we use the conventional Lebesgue measure).

Next, we claim that for all $x \in [0,1]$, we have $x \in X_i$ for some $i \in \mathbb{N}$. Let $x \in [0,1]$ be arbitrary. Since $\sim$ partitions $[0,1]$, we have x is in one equivalence class. Let y be the representative of that class that gets chosen into X. By definition of $\sim$, we have $x - y$ is a rational number. Since $x, y \in [0,1]$, we have that rational is in $[-1,1]$. Let us denote this rational r_i. Since $r_i \in \mathbb{Q} \cap [-1,1]$ and $y + r_i = x$, thus $x \in X_i$.

Thus, have $[0,1] \subseteq \bigcup\{X_i\}_{i \in \mathbb{N}}$. Since the measure of $[0,1]$ is 1, we have the measure of $\bigcup\{X_i\}_{i \in \mathbb{N}}$ is larger or equal to 1. Let μ denote the Lebesgue measure, we have $1 \leq \mu(\bigcup\{X_i\}_{i \in \mathbb{N}}) \leq 3$.

Finally, we claim that X_i and X_j are disjoint for any $i \neq j$. Suppose to the contrary that they are not. Then we can fix $a, b \in X$ and r_i, r_j with $a + r_i = b + r_j$. And thus we have $a - b = r_j - r_i \in \mathbb{Q}$, which is a contradiction, because a and b should differ by an irrational number, since they are from two equivalence classes. Thus X_i and X_j are disjoint.

Thus, we have

$$\mu\left(\bigcup\{X_i\}_{i\in\mathbb{N}}\right) = \sum \mu(X_i) \tag{1}$$

. Also, since the action to construct X_i from X is a simple shifting, which is isometric, we have $\mu(X) = \mu(X_1) = \mu(X_2) = \cdots$ Thus, we have

$$\mu\left(\bigcup\{X_i\}_{i\in\mathbb{N}}\right) = \sum \mu(X) \tag{2}$$

. Now, since $\mu(\bigcup\{X_i\}_{i\in\mathbb{N}}) \geq 1$, we have $\mu(X) \neq 0$. However, if $\mu(X) > 0$, then $\sum \mu(X) = \infty$. But since $\mu(\bigcup\{X_i\}_{i\in\mathbb{N}}) \leq 3$, this also cannot be true. Thus the set X is non-measurable.

$\square$

The proof above looks a bit long but is very easy to follow. The technique/trick mathematicians use for proving the aforementioned paradoxes is very similar to the technique used in this proof: basically you just evoke the Axiom of Choice to construct several non-measurable sets out of some "nice" sets (the sets whose measure is conventionally well-defined), and then show some paradoxical outcomes regarding to the volume of these non-measurable sets. To get some sense of what I am talking about, let us directly look at the content of the three main paradoxes, i.e., the Hausdorff paradox, the Banach-Tarski paradox and the Von Neumann paradox. But first, let us define some notations.

Notation 3 (Disjoint Union). *Let X, Y be sets. We write $Z = X \dot\cup Y$ to denote that the set Z is the union of X and Y and further that X and Y are disjoint.*

Notation 4 (Congruence). *Let $X, Y \subseteq \mathbb{R}^3$. We write $X \cong Y$ to denote that X and Y are congruent, i.e., X can be obtained from Y by isometric transformations (translation, rotation and reflection), and vice versa.*

Notation 5 (Equivalence by finite decomposition). *Let X, Y be sets. We write $X \overset{n}{=} Y$ to denote that X and Y are equivalent by finite decomposition, i.e., there are disjoint decompositions of X and Y into n sets*

$$X = X_1 \dot\cup X_2 \dot\cup \cdots \dot\cup X_n, Y = Y_1 \dot\cup Y_2 \dot\cup \cdots \dot\cup Y_n$$

with $X_i \cong Y_i$ for $1 \leq i \leq n$.

Now we are ready to state the content of the three famous paradoxes. The Hausdorff paradox was introduced and proved by Felix Hausdorff in 1914.

Theorem 6 (Hausdorff Paradox). *Let S be a sphere in $\mathbb{R}^3$, i.e. S is the surface of a ball in $\mathbb{R}^3$. There exists a partition of S into 4 disjoint sets $S = A \dot\cup B \dot\cup C \dot\cup D$ such that D is a countable set, and $A \cong B, A \cong C$ and $A \cong B \dot\cup C$.*

It turns out that the sets A, B, C above are all non-measurable sets, since if otherwise, then the measure of A will be equal to the measure of B and be equal to two times the measure of B at the same time. We will sketch out the proof for Hausdorff paradox later in this section. Now, assuming the Hausdorff paradox, the Banach-Tarski paradox almost follows naturally (provided some mathematical intuition).

Theorem 7 (Banach-Tarski Paradox). *Let A, B, C be three balls in $\mathbb{R}^3$ with $A \cong B \cong C$. There exist a partition of B into n disjoint pieces, a partition of C into m disjoint pieces and a partition of A into $n+m$ disjoint pieces*

$$B = B_1 \dot{\cup} B_2 \dot{\cup} \cdots \dot{\cup} B_n, C = C_1 \dot{\cup} C_2 \dot{\cup} \cdots \dot{\cup} C_m, A = A_1 \dot{\cup} A_2 \dot{\cup} \cdots \dot{\cup} A_{n+m}$$

such that $B_i \cong A_i$ for $1 \leq i \leq n$ and $C_i \cong A_{n+i}$ for $1 \leq i \leq m$.

Again, some A_i, B_i, C_i are non-measurable. This paradox basically says that we can decompose a ball into finitely many pieces and put them back together to form two new balls which have the same volume as the original ball. Actually, this "finitely many" can be as small as five. The Banach-Tarski paradox was proposed in 1924. After about five years, Von Neumann proved another even more powerful paradox.

Theorem 8 (Von Neumann Paradox). *Let A be a square in $\mathbb{R}^2$. Let $B \subsetneq A$ be a smaller square that is a strict subset of A. We have $A \stackrel{n}{=} B$ for some finite n.*

The reason why Von Neumann paradox is more powerful is that it extends the paradoxical decomposition to 2D planes. But the proving strategy is similar to the one used for the Hausdorff paradox. Now we will sketch the proof idea for the Hausdorff paradox.

3.1 Sketch of proof for the Hausdorff paradox

Sketch of proof. Let S be a sphere in $\mathbb{R}^3$ centered at the origin O. Let P, Q be two points on S. Let $\sigma \in M_3(\mathbb{R})$ (i.e., σ is a 3 by 3 matrix) be the rotation matrix that rotates points by π around axis OP, and let $\tau \in M_3(\mathbb{R})$ be the rotation matrix that rotates points by $\dfrac{2\pi}{3}$ around axis OQ. The reason why we chose these two specific angles will be apparent later. Let G be the group of 3 by 3 matrices generated by σ and τ, i.e.,

$$G = \langle \sigma, \tau \rangle \tag{3}$$

Note that $\sigma^2 = I$ and $\tau^3 = I$. Thus σ, τ, τ^2 are three distinct rotations. Thus any $g \in G$ can be expressed as a sequence of rotations

$$g = \sigma \tau^{a_1} \sigma \tau^{a_2} \sigma \tau^{a_3} \cdots \tag{4}$$

or

$$g = \tau^{a_1} \sigma \tau^{a_2} \sigma \tau^{a_3} \sigma \cdots \tag{5}$$

with $a_i \in \{1, 2\}$. Astute readers might note that some such sequences of rations are not distinct. In particular, some $g \in G$ might be equivalent to I. However, we can show that for there to be such $g \in G$, the angle between OP and OQ has to be chosen very specifically. In fact, there are only finitely many values of the angle that can result in such g. Since the number of axis is infinite, we can choose P and Q so that the angle between OP and OQ avoids those values. Thus we have that two sequences of rations are distinct unless every one of their terms is the same.

It is well-known that any composition of rotations is a rotation. Thus any $g \in G$ is a rotation about some axis through the origin. Let us use "poles" to call the two points at the intersection between an axis and S. Since G is a countable set, there are countably many axis, and thus there are countably many poles. Now we take out all the poles in S and form a countable set D. The reason why we take these points out will be clarified later.

Now note that G acts on $S \setminus D$ via matrix-vector multiplication, i.e. we can rotate the points in $S \setminus D$ using any $g \in G$. Let $*$ denote such action. The **orbit** of any $s \in S \setminus D$ is define to be the set

$$O_s = \{g * s : g \in G\} \tag{6}$$

, i.e., all the points in $S \setminus D$ that can be reached from s via rotations in G. In fact, the relation "differ by some $g \in G$" is an equivalence relation on $S \setminus D$ and the orbits are the equivalence classes. Thus, we have the orbits partition $S \setminus D$, that is, we can write

$$S \setminus D = O_1 \dot{\cup} O_2 \dot{\cup} O_3 \cdots \tag{7}$$

where O_i is an orbit.

Now we can evoke the Axiom of Choice, and choose one representative from each orbit and form a new set $X \subseteq S \setminus D$ with $X = \{o_i\}$ where $o_i \in O_i$ for all i. Let gX be defined as the set

$$gX = \{g * x : x \in X\} \tag{8}$$

for any $g \in G$. We can easily show that for any two distinct $g_1, g_2 \in G$, we have g_1X and g_2X are disjoint. In particular, gX and X are disjoint for any $g \in G$ because we have taken out all the poles in S.

Also, we can easily show that

$$\bigcup_{g \in G} gX = S \setminus D \tag{9}$$

. That is, by rotating X using every $g \in G$ and union together the results, we can get back $S \setminus D$.

Now note that the elements of G can be partitioned into three sets: G_σ that contains the sequences that starts with σ, G_τ that contains the sequences that starts with τ and G_{τ^2} that contains the sequences that starts with τ^2.

Now we can partition $S \setminus D$ into three disjoint sets A, B, C using $G_\sigma, G_\tau, G_{\tau^2}$: $A = \bigcup_{g \in G_\sigma} gX$, $B = \bigcup_{g \in G_\tau} gX$ and $C = \bigcup_{g \in G_{\tau^2}} gX$. We have $S \setminus D = A \dot{\cup} B \dot{\cup} C$.

Now note that for any $g \in G_\sigma$, we have $\sigma \circ g \in G_\tau \cup G_{\tau^2}$, since g starts with σ, and composing one more σ at the front cancels it out since $\sigma^2 = I$. The term after the original first σ must be τ or τ^2. Thus $\sigma \circ g \in G_\tau \cup G_{\tau^2}$. Also, for any $g \in G_\tau \cup G_{\tau^2}$, taking away the first term will result in a rotation in G_σ. Thus we have

$$\sigma G_\sigma = G_\tau \cup G_{\tau^2} \tag{10}$$

, where $\sigma G_\sigma = \{\sigma \circ g : g \in G_\sigma\}$.

By the same reasoning, we have $\tau G_\sigma = G_\tau$ and $\tau^2 G_\sigma = G_{\tau^2}$.

Thus, we can easily show that $\sigma A = B \cup C$, $\tau A = B$ and $\tau^2 A = C$. Thus, $A \cong B$, $A \cong C$ and $A \cong B \cup C$ since σ, τ, τ^2 are isometric transformations.

Now we have $S = A \dot{\cup} B \dot{\cup} C \dot{\cup} D$ where D is countable, $A \cong B$, $A \cong C$ and $A \cong B \cup C$ as desired. $\square$

References

[1] EC Milner. HAUSDORFF'S PARADOX - OR TWO FOR ONE.

[2] Eric Schechter. A home page for the AXIOM OF CHOICE.